

FERMIONS ON THE LIGHT-FRONT

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Abstract

Issues that are specific for formulating fermions in light-cone quantization are discussed. Special emphasis is put on the use of parity invariance in the non-perturbative renormalization of light-cone Hamiltonians.

Light-front (LF) quantization is the most physical approach to calculating parton distributions on the basis of QCD¹. Before one can formulate QCD with quarks, it is necessary that one understands how to describe fermions in this framework. This in turn requires that one addresses the following issues

- How is spontaneous symmetry breaking (chiral symmetry!) manifested in the LF framework, where the vacuum appears to be trivial?
- Is it possible to preserve current conservation and parity invariance in this framework?
- How does one formulate fermions on the transverse lattice, which seems to be a very promising approaches to pure glue LFQCD^{2,3?}^a

1 Spontaneous Symmetry Breaking

The first of the above issues has been addressed very often in the past and we will restrict ourselves here to a brief summary.^b In the LF framework, non-trivial vacuum structure can reside only in zero-modes ($k^+ = 0$ modes). Since these are high-energy modes (actually infinite energy in the continuum) one often does not include them as explicit degrees of freedom but assumes they have been integrated out, leaving behind an effective LF-Hamiltonian. The important points here are the following. If the zero-mode sector involves spontaneous symmetry breaking, this manifests itself as explicit symmetry breaking for the effective Hamiltonian. In general, these effective LF Hamiltonians thus have a much richer operator structure than the canonical Hamiltonian. Therefore, compared to a conventional Hamiltonian framework, the question of the vacuum has been shifted from the states to the operators and it should thus be clear that the issues of renormalization and the vacuum are deeply entangled in the LF framework.

^aBecause of lack of space, this important issue could not be discussed in these notes.

^bSee for example Ref. ¹ for a more extensive discussion on this question.

2 Current Conservation

Despite widespread confusion on this subject, vector current conservation (VCC) is actually not a problem in the LF framework. Many researchers avoid the subject of current conservation because the divergence of the vector current

$$q_\mu j^\mu(q) = q^- j^+ + q^+ j^- - \vec{q}_\perp \vec{j}_\perp \quad (1)$$

involves j^- , which is quadratic in the constrained fermion spinor component

$$j^- = \psi_{(-)}^\dagger \psi_{(-)}, \quad (2)$$

and thus j^- contains quartic interactions making it at least as difficult to renormalize as the Hamiltonian.

We know already from renormalizing P^- that the canonical relation between $\psi_{(-)}$ and $\psi_{(+)}$ is in general not preserved in composite operators (such as $\bar{\psi}\psi$). It is therefore clear that a canonical definition for j^- will in general violate current conservation since it does not take into account integrating out zero-modes and other high-energy degrees of freedom.

However, in LF gauge $A^+ = 0$, j^- does not enter the Hamiltonian and therefore one can address its definition separately from the construction of the Hamiltonian. In fact, it is very easy to find a pragmatic definition which obviously guarantees manifest current conservation, namely

$$j^-(q^+) = -\frac{1}{q^+} [P^-, j^+(q^+)] \quad (1+1) \quad (3)$$

$$j^-(q^+, \vec{q}_\perp) = -\frac{1}{q^+} \left\{ [P^-, j^+(q^+, \vec{q}_\perp)] - \vec{q}_\perp \vec{j}_\perp(q^+, \vec{q}_\perp) \right\} \quad (3+1)$$

in 1+1 and 3+1 dimensions respectively. The corresponding expressions in coordinate space (x^-, \vec{x}_\perp) can be obtained by Fourier transform. In summary,

- Most importantly, VCC is no problem in LF quantization and is manifest at the operator level, provided j^- is *defined* using Eq. (3).
- Since VCC can easily be made manifest (by using the above construction!), there is no point in *testing* its validity and it cannot be used as a non-perturbative renormalization condition either.
- For a non-interacting theory, Eq. (3) reduces to the canonical definition of j^- , but in general this is not the case when P^- contains interactions or even non-canonical terms.

Note that (as so often on the LF) q^+ appears in the denominator of Eq. (3). Therefore, as usual, one should be very careful while taking the $q^+ \rightarrow 0$ limit and while drawing any conclusions about this limit. An example of this kind are the pair creation terms in j^- . Naively they do not contribute for $q^+ \rightarrow 0$, since the $q\bar{q}$ pair emanating from j^- necessarily carries positive q^+ . However, since j^- often has very singular matrix elements for q^+ , such seemingly vanishing terms nevertheless survive the $q^+ \rightarrow 0$ limit, which often leads to confusion. For an early example of this kind see Ref. ⁴. A more recent discussion can be found in Ref. ⁵.

3 Parity Invariance

3.1 General Remarks

A parity transformation, $x^0 \xrightarrow{P} x^0$, $\vec{x} \xrightarrow{P} -\vec{x}$ leaves the quantization hyperplane ($x^0 = 0$) in equal time (ET) quantization invariant and therefore the parity operator is a kinematic operator in such a framework. It is thus very easy to ensure that parity is a manifest symmetry in ET quantization by tracking parity at each step in a calculation. The situation is completely different in the LF framework, where the same parity transformation exchanges LF-‘time’ ($x^+ \equiv x^0 + x^3$) and space ($x^- \equiv x^0 - x^3$) directions, i.e.

$$x^+ \xleftrightarrow{P} x^- \quad (4)$$

and therefore the quantization hyperplane $x^+ = 0$ is *not* invariant. Hence, the parity operator is a dynamical operator on the LF and, except for a free field theory, it is probably impossible to write down a simple expression for it in terms of quark and gluon field operators. Thus, parity invariance is not a manifest symmetry in this framework. Note that the situation is the other way round for the boost operator (kinematic and manifest on the LF, dynamical and non-manifest in ET). It thus depends on the physics application one is interested in and the symmetries that one considers the most important ones for that particular physics problem, which framework is preferable.

For most applications of LF quantization, the lack of manifest parity actually does not constitute a problem — in fact, one can view it as an opportunity rather than a problem. The important point here is the following: due to the lack of manifest covariance in a Hamiltonian formulation, LF Hamiltonians in general contain more parameters than the corresponding Lagrangian. Parity invariance may be very sensitive to some of these parameters.

In order to illustrate this important point, let us consider the example of

a 1+1 dimensional Yukawa model⁶

$$\mathcal{L} = \bar{\psi} (i \not{\partial} - m_F - g\phi\gamma_5) \psi - \frac{1}{2} \phi (\square + m_B^2) \phi. \quad (5)$$

This model actually has a lot in common with the kind of interactions that appear when one formulates QCD (with fermions) on a transverse lattice⁷.

The main difference between scalar and Dirac fields in the LF formulation is that not all components of the Dirac field are dynamical: multiplying the Dirac equation

$$(i \not{\partial} - m_F - g\phi\gamma_5) \psi = 0 \quad (6)$$

by $\frac{1}{2}\gamma^+$ yields a constraint equation (i.e. an “equation of motion” without a time derivative)

$$i\partial_- \psi_{(-)} = (m_F + g\phi\gamma_5) \gamma^+ \psi_{(+)}, \quad (7)$$

where $\psi_{\pm} \equiv \frac{1}{2}\gamma^{\mp}\gamma^{\pm}\psi$. For the quantization procedure, it is convenient to eliminate $\psi_{(-)}$, using

$$\psi_{(-)} = \frac{\gamma^+}{2i\partial_-} (m_F + g\phi\gamma_5) \psi_{(+)} \quad (8)$$

from the classical Lagrangian before imposing quantization conditions, yielding

$$\begin{aligned} \mathcal{L} = & \sqrt{2}\psi_{(+)}^{\dagger} i\partial_+ \psi_{(+)} - \phi (\square + m_B^2) \phi - \psi_{(+)}^{\dagger} \frac{m_F^2}{\sqrt{2}i\partial_-} \psi_{(+)} \\ & - \psi_{(+)}^{\dagger} \left(g\phi \frac{m_F\gamma_5}{\sqrt{2}i\partial_-} + \frac{m_F\gamma_5}{\sqrt{2}i\partial_-} g\phi \right) \psi_{(+)} - \psi_{(+)}^{\dagger} g\phi \frac{1}{\sqrt{2}i\partial_-} g\phi \psi_{(+)}. \end{aligned} \quad (9)$$

The rest of the quantization procedure very much resembles the procedure for self-interacting scalar fields.

The above canonical Hamiltonian contains a kinetic term for the fermions, a fermion boson vertex and a fermion 2-boson vertex. While the couplings of these three terms in the canonical Hamiltonian depend only on two independent parameters (m and g), it turns out that these terms are renormalized independently from each other once zero-mode and other high-energy degrees of freedom are integrated out. More explicitly this means that one should make an ansatz for the renormalized LF Hamiltonian density of the form

$$\begin{aligned} \mathcal{P}^- = & \frac{m_B^2}{2} \phi^2 + \psi_{(+)}^{\dagger} \frac{c_2}{\sqrt{2}i\partial_-} \psi_{(+)} + c_3 \psi_{(+)}^{\dagger} \left(\phi \frac{\gamma_5}{\sqrt{2}i\partial_-} + \frac{\gamma_5}{\sqrt{2}i\partial_-} g\phi \right) \psi_{(+)} \\ & + c_4 \psi_{(+)}^{\dagger} \phi \frac{1}{\sqrt{2}i\partial_-} \phi \psi_{(+)}, \end{aligned} \quad (10)$$

where the c_i do not necessarily satisfy the canonical relation $c_3^2 = c_2 c_4$. However, this does not mean that the c_i are completely independent from each other. In fact, Eq.(10) will describe the Yukawa model only for specific combinations of c_i . It is only that we do not know the relation between the c_i .

Thus the bad new is that the number of parameters in the LF Hamiltonian has increased by one (compared to the Lagrangian). The good news is that a wrong combination of c_i will in general give rise to a parity violating theory: formally this can be seen in the weak coupling limit, where the correct relation ($c_3^2 = c_2 c_4$) follows from a covariant Lagrangian. Any deviation from this relation can be described on the level of the Lagrangian (for free massive fields, equivalence between LF and covariant formulation is not an issue) by addition of a term of the form $\delta\mathcal{L} = \bar{\psi} \frac{\gamma^+}{i\partial^+} \psi$, which is obviously parity violating, since parity transformations result in $A^\pm \xrightarrow{P} A^\mp$ for Lorentz vectors A^μ ; i.e. $\delta\mathcal{L} \xrightarrow{P} \bar{\psi} \frac{\gamma^-}{i\partial^-} \psi \neq \delta\mathcal{L}$. This also affects physical observables, as can be seen by considering boson fermion scattering in the weak coupling limit of the Yukawa model. At the tree level, there is an instantaneous contact interaction, which is proportional to $\frac{1}{q^+}$. The (unphysical) singularity at $q^+ = 0$ is canceled by a term with fermion intermediate states, which contributes (near the pole) with an amplitude $\propto -\frac{m_v^2}{m_{kin}^2} \frac{1}{q^+}$. Obviously, the singularity cancels iff $m_V = m_{kin}$. Since the singularity involves the LF component q^+ , this singular piece obviously changes under parity. This result is consistent with the fact that there is no zero-mode induced renormalization of m_{kin} and thus $m_V = m_{kin}$ at the tree level. This simple example clearly demonstrates that a ‘false’ combination of m_V and m_{kin} leads to violations of parity for a physical observable, which is why imposing parity invariance as a renormalization condition may help reduce the dimensionality of coupling constant space.

3.2 Parity Sensitive Observables that “don’t work”

Of course there are an infinite number of parity sensitive observables, but not all of them are easy to evaluate non-perturbatively in the Hamiltonian LF framework. Furthermore, we will see below that some relations among observables, which seem to be sensitive to parity violations, are actually ‘protected’ by manifest symmetries, such as charge conjugation, or by VCC.

From the brief discussion of the (to-be-canceled) singularity above it seems that the most sensitive observable to look for violations of parity in QCD

^cCoupling coherence⁸ does not help to determine the finite part (integration constant) for the mass if the original Lagrangian contains a nonzero bare mass!

would be Compton scattering cross sections between quarks and gluons (or the corresponding fermions and bosons in other field theories) because there one could tune the external momenta such that the potential singularity enters with maximum strength. However, this is not a very good choice: first of all non-perturbative scattering amplitudes are somewhat complicated to construct on the LF. Secondly, quarks and gluons are confined particles, which makes qg Compton scattering an unphysical process.

A much better choice are any matrix elements between bound states. Bound states are non-perturbative and all possible momentum transfers occur in their time evolution. Therefore, any parity violating sub-amplitude would contribute at some point and would therefore affect physical observables. Secondly, since one of the primary goal of LFQCD is to explore the non-perturbative spectra and structure of hadrons, matrix elements in bound states are the kind of observables for which the whole framework has been tailored.

One conceivable set of matrix elements are those of the vector current operator j^\pm . For example, consider the vacuum to meson matrix elements

$$\begin{aligned}\langle 0|j^+|n,p\rangle &= p^+ f_n^{(+)} \\ \langle 0|j^-|n,p\rangle &= p^- f_n^{(-)}.\end{aligned}\tag{11}$$

By boost invariance, the couplings defined in Eq. (11) must be independent of the momenta. Obviously, parity invariance requires $|f_n^{(+)}| = |f_n^{(-)}|$. However, this relation also follows from current conservation $0 = p^- \langle 0|j^+|n,p\rangle + p^+ \langle 0|j^-|n,p\rangle$, which makes it a useless relation for the purpose of parity tests.

Similar statements can be made about elastic formfactors but we will omit the details here. The basic upshot is that the same relations between the matrix elements of j^\pm that arise from parity invariance can often also be derived using only VCC, i.e. such relations are in general ‘protected’ by VCC

One may also consider non-conserved currents, such as the axial vector current. However, there one would have to face the issue of defining the ‘minus’ component before one can test any parity relations. Parity constraints are probably very helpful in this case when constructing the minus components, but then those relations can no longer be used to help constrain the coefficients in the LF-Hamiltonian.

Another class of potentially useful operators consists of the scalar and pseudoscalar densities $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$. Obviously, if parity is conserved then at most one of the two couplings

$$\begin{aligned}f_S &= \langle 0|\bar{\psi}\psi|n\rangle \\ f_P &= \langle 0|\bar{\psi}\gamma_5\psi|n\rangle\end{aligned}\tag{12}$$

can be nonzero at the same time, since a state $|n\rangle$ cannot be both scalar and pseudoscalar. What restricts the usefulness of this criterion is the fact that the same ‘selection rule’ also follows from charge conjugation invariance ($\bar{\psi}$ and $\bar{\psi}\gamma_5\psi$ have opposite charge parity!) which is a manifest symmetry on the LF. Therefore, only for theories with two or more flavors, where one can consider operators such as $\bar{u}s$ and $\bar{u}\gamma_5s$, does one obtain parity constraints that are not protected by charge conjugation invariance. But even then one may have to face the issue of how to define these operators. Parity may of course be used in this process, but then it has again been ‘used up’ and one can no longer use these selection rules to test the Hamiltonian.

In summary, many parity relations are probably very useful to determine the LF representation of the operators involved, but may not be very useful to determine the Hamiltonian.

3.3 A useful observable to test parity

We have seen above that relations between j^+ and j^- are often protected by vector current conservation and are therefore not very useful as parity tests. A much more useful parity test can actually be obtained by considering the ‘plus’-component only: Let us now consider a vector form factor,

$$\langle p', n | j^\mu | p, m \rangle \stackrel{!}{=} \varepsilon^{\mu\nu} q_\nu F_{mn}(q^2), \quad (13)$$

where $q = p' - p$, between states of opposite parity. When writing the r.h.s. in terms of one invariant form factor, use was made of both vector current conservation and parity invariance. A term proportional to $p^\mu + p'^\mu$ would also satisfy current conservation, but has the wrong parity. A term proportional to $\varepsilon^{\mu\nu} (p_\nu + p'_\nu)$ has the right parity, but is not conserved and a term proportional to q^μ is both not conserved and violates parity. Other vectors do not exist for this example. The Lorentz structure in Eq. (13) has nontrivial implications even if we consider only the “plus” component, yielding

$$\frac{1}{q^+} \langle p', n | j^+ | p, m \rangle = F_{mn}(q^2). \quad (14)$$

That this equation implies nontrivial constraints can be seen as follows: as a function of the longitudinal momentum transfer fraction $x \equiv q^+/p^+$, the invariant momentum transfer reads (M_m^2 and M_n^2 are the invariant masses of the in and outgoing meson)

$$q^2 = x \left(M_m^2 - \frac{M_n^2}{1-x} \right). \quad (15)$$

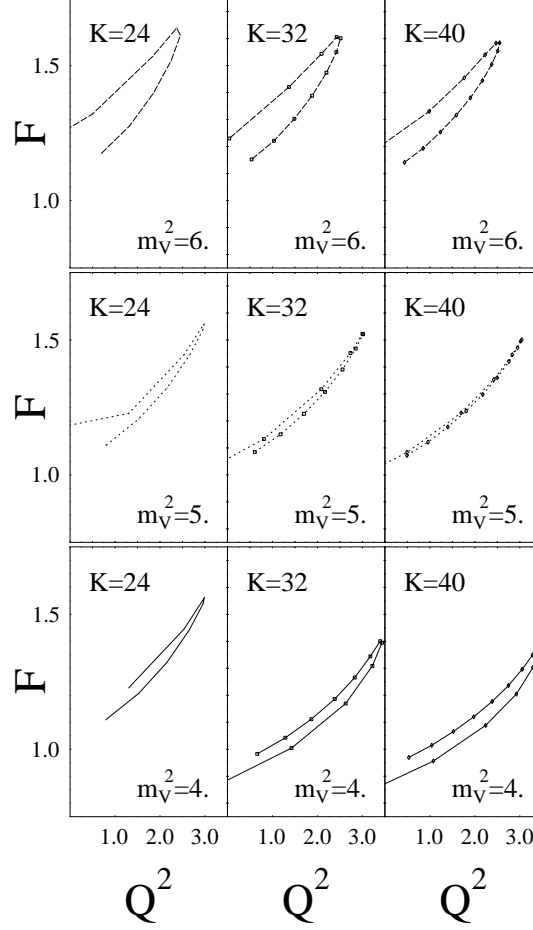


Figure 1: Inelastic transition form factor (14) between the two lightest meson states of the Yukawa model, calculated for various vertex masses m_v and for various DLCQ parameters K . The physical masses for the fermion and the scalar meson have been renormalized to the values $(m_F^{phys})^2 = (m_S^{phys})^2 = 4$. All masses and momenta are in units of $\sqrt{\lambda} = \sqrt{c_4/2\pi}$. In this example, only for $m_V^2 \approx 5$ one obtains a form factor that is a unique function of Q^2 , i.e. only for $m_V^2 \approx 5$, the result is consistent with Eq. (14). Therefore, only for this particular value of the vertex mass, is the matrix element of the current operator consistent with both parity and current conservation.

This quadratic equation has in general, for a given value of q^2 , two solutions for x which physically correspond to hitting the meson from the left and right respectively. The important point is that it is not manifestly true that these two values of x in Eq. (14) will give the same value for the form factor, which makes this an excellent parity test.

In Ref. ⁶, the coupling as well as the physical masses of both the fermion and the lightest boson were kept fixed, while the “vertex mass” was tuned (note that this required re-adjusting the bare kinetic masses). Figure 1 shows a typical example, where the calculation of the form factor was repeated for three values of the DLCQ parameter K (24, 32 and 40) in order to make sure that numerical approximations did not introduce parity violating artifacts.

For a given physical mass and boson-fermion coupling, there exists a “magic value” of the vertex mass and only for this value one finds that the parity condition (14) is satisfied over the whole range of q^2 considered. This provides a strong self-consistency check, since there is only one free parameter, but the parity condition is not just one condition but a condition for every single value of q^2 (i.e. an infinite number of conditions). In other words, keeping the vertex mass independent from the kinetic mass is not only necessary, but also seems sufficient in order to properly renormalize $Yukawa_{1+1}$.

In the above calculation, it was sufficient to work with a vertex mass that was just a constant. However, depending on the interactions and the cutoffs employed, it may be necessary to introduce counter-term functions⁸. But even in cases where counter-term functions need to be introduced, parity constraints may be very helpful in determining the coefficient-functions for those more complex effective LF Hamiltonians non-perturbatively.

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